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Digital Simulation of A White Noise Model Formed of Uniformly Almost Periodic Functions

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Starting from a criticism of the Rice representations of normal random noise, a new mathematical model of white noise expressed in terms of uniformly almost periodic functions is obtained. By means of digital simulation it is confirmed that the model exhibits white noise characteristics.

1. INTRODUCTORY REMARKS

There are many examples of random processes in nature, one of which is stationary random noise in electric circuits caused by the thermal agitation or shot effect of electrons. If the behavior of an electron in such a noise current is closely observed, it will be found that the behavior itself is governed by the so-called law of cause and effect, but is accidental to that of other electrons. The noise current, i.e., the totality of electrons, each of which behaves according to the law of causality but accidentally to other electrons, exhibits regularities known as statistical properties, and it is for these that a mathematical theory can be constructed.

Therefore, when a mathematical model of random noise is sought, what is necessary is first to define a suitable deterministic expression to describe a sequence of random phenomena, e.g., the movement of electrons, and then to incorporate the concept of probability distribution which accounts for the above-mentioned "accidental character" into the mathematical framework. An important problem is how to unify the deterministic character and the probabilistic or stochastic character of random noise. To be specific, what we must consider are

(1) The type of deterministic expression (functions of time) to be used;

(2) The kind of probability distribution to be chosen;

(3) How to incorporate the probability distribution into the deterministic expression.

Important examples of deterministic expressions are the differential equations of Langevin and Fokker–Planck (Chandrasekhar, 1943). The stochastic properties are embodied in their terms of external force. If an orthonormal series expansion (e.g., Fourier series in the Rice representations (Rice, 1944) or sampling functions in the Shannon representation) is used as the functions of time, the probabilistic characteristics are expressed in the expansion coefficients.

It should be noted that a deterministic expression which lays a foundation for a mathematical model of random noise is directly related to a random process which really exists in the physical world, while the probability distribution expresses a possibility that the random process can assume any value in a specified interval. From this point of view it is obvious that the usefulness of mathematical models of random noise is not limited to the problem of electrical noise like shot noise or thermal noise, but has wide applications for other fields of engineering and physics, e.g., external disturbances in feedback control systems, fading, turbulent flow of gas, Brownian motion, etc.

In this paper it is shown that a mathematical model of random noise can be formed in terms of a trigonometric series consisting of uniformly almost periodic functions (Kawada, 1952). The process of formation of white noise is simulated by means of a digital computer, and it is thereby confirmed that the mathematical model exhibits white noise characteristics when the number of terms becomes sufficiently large. In view of the complexity of the problem, the use of a digital computer is inevitable for experimental confirmation.

2. A Critique of the Rice Representations of Random Noise

S. O. Rice has given the well-known representations of normal random noise,

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right), \tag{1}$$

$$I(t) = \sum_{n=1}^{N} c_n' \cos\left(\frac{2\pi}{T} nt + \varphi_n\right), \tag{2}$$

where the coefficients a_n and b_n are independent random variables distributed normally with zero means, whereas c_n' are constants. φ_n are uniformly distributed over an interval $[0, 2\pi]$, and N and T are assumed to be large.

Let us begin with the physical interpretation of Eq. (1). Suppose that a time-limited record of random noise is divided into N nonoverlapping parts of equal length of T seconds and that all of the random noise records appearing in the second through N-th intervals are projected on the first interval. Thus, we have an ensemble of N distinct random noise records in the first interval. A set of possible amplitudes at time t from the ensemble of random noise records in the first interval may be expected to follow a normal distribution for large N. The normal distribution is a consequence of the central limit theorem. This idea helps justify the first Rice representation of normal random noise (see Eq. (1)), i.e., a Fourier series representation of random noise whose coefficients are normally distributed with fixed variances.

In the second Rice representation of random noise (see Eq. (2)) the expansion coefficients c_n' are regarded as constants but φ_n are uniformly distributed over the interval $[0, 2\pi]$. According to the central limit theorem $I(t)$ is asymptotically normal as N tends to infinity.

There is one point that needs to be mentioned in connection with the Rice representations of random noise. This is to the effect that the probability distributions which a_n, b_n, or φ_n are assumed to follow in no way describe the random variation itself of amplitude or phase in the course of time but express possible variation of amplitude or phase at a fixed time t. The truth is that in random processes existing in the physical world all possible variations of amplitude, phase, or other physical quantities really appear in a sufficiently long interval of time. In view of the above discussion, the Rice representations of random noise do not seem to yield an organic unity of the deterministic character and stochastic character of random noise, since the probability distributions are assumed at the outset independently of the lapse of time.

3. Formation of White Noise Model by Means of Uniformly Almost Periodic Functions

In Eqs. (1) and (2) the ratio of one fundamental frequency to another is always a reciprocal of an integer. This seems to be the very reason why the probability distributions have to be assumed independently of the lapse of time in the Rice representations. In this section a new mathematical model of random noise will be proposed, into which it is not necessary to introduce any probability distribution law at the outset because a probability distribution is automatically formed within the course of time. Let us express random noise in terms of trigonometric functions as

$$I_N(t) = \sum_{n=1}^{N} I_n(t) = \sum_{n=1}^{N} c_n \cos 2\pi(f_n t + \phi_n), \tag{3}$$

where frequency ratios such as f_1/f_2, f_2/f_3,..., form a set of irrational numbers (Ohta, 1956). Here $I_N(t)$ is not strictly periodic, but uniformly almost periodic in a sense that for each positive number ϵ there exist dense numbers T such that (Kawada, 1952)

$$\sup_{-\infty < t < \infty} | I_N(t + T) - I_N(t)| \leqslant \epsilon.$$

The set of all positive rational numbers is enumerable. The set of all irrational numbers is much denser than the above set. When all of the frequency ratios are irrational numbers, from the quasiergodic hypothesis of P. and T. Ehrenfest it can be shown that within a finite interval of time, $I_N(t)$ of Eq. (3) can go through a neighborhood of an arbitrary point in a volume enclosed by hyperplanes $x_1 = \pm c_1$, $x_2 = \pm c_2$,..., $x_N = \pm c_N$ in an N-dimensional topological space. Each point in the N-dimensional space corresponds to a possible state of $I_N(t)$. This implies that $I_N(t)$ is capable of expressing random properties of an actual random noise. The above discussion helps justify using the uniformly almost periodic function $I_N(t)$ as a mathematical model of white noise.

4. Normal Distribution Characteristic of the New White Noise Model

It must be noted that we have not assumed any type of probability distribution for ϕ_n in Eq. (3). But it is true that $I_N(t)$ is expected to follow a normal distribution when N approaches infinity. Let us prove this.

In Eq. (3) the sum of two sinusoidal functions, $c_i \cos 2\pi(f_i t + \phi_i)$ and $c_k \cos 2\pi(f_k t + \phi_k)$, is a uniformly almost periodic function (hereafter it will be abbreviated as u.a.p. function), when f_k/f_i is an irrational number. The sum of u.a.p. functions is also a u.a.p. function. Therefore $I_N(t)$ is a u.a.p. function. Suppose that a distribution function defined as

$$\delta_N(I)_T = \frac{1}{2T} mE(t; I_N(t) \leqslant I, -T \leqslant t \leqslant T) \tag{4}$$

is convergent and that it has an asymptotic distribution function $\delta_N(I)$. Here mE denotes the Lebesgue measure of Borel set E. From some well-known properties of the u.a.p. function (Kawada, 1952) it follows that $I_N(t)$ has the asymptotic distribution $\delta_N(I)$. It also can be shown that $I_N(t)$ possesses a probability distribution density $P(I) = d\delta_N(I)/dI$ when $\delta_N(I)$ is absolutely continuous. This $\delta_N(I)$ is uniquely determined by the following characteristic function:

$$M(e^{iuI_N(t)}) = \int_{-\infty}^{\infty} e^{iuI}\, d\delta_N(I). \tag{5}$$

On the basis of the above discussion it can be shown from the Kronecker–Weyl theorem that $I_N(t)$, which may be regarded as a vector in an N-dimensional Euclidean space, has an asymptotic distribution function,

$$\delta_N(I) = \beta_1(I) * \beta_2(I) * \cdots * \beta_N(I), \tag{6}$$

where $\beta_n(I)$ is an asymptotic distribution function for $I_n(t)$ given as

$$\beta_n(I) = \begin{cases} 0 & (I < -c_n), \\ 1 - \dfrac{1}{\pi} \arc\cos \dfrac{I}{c_n} & (-c_n \leqslant I \leqslant c_n), \\ 1 \quad (I > c_n), & (n = 1, 2, \ldots, N). \end{cases} \tag{7}$$

The symbol $*$ in the above denotes the convolution integral (Kawada, 1952). The Fourier–Stieltjes transform of $\delta_N(I)$ in Eq. (6) is given by

$$\prod_{n=1}^{N} J_0(c_n u). \tag{8}$$

Obviously, the Fourier–Stieltjes transform of $\delta_N(I)$ becomes $\prod_{n=1}^{\infty} J_0(c_n u)$ when N goes to infinity. Here c_n are assumed equal to a constant c_0 which is independent of n. Hence, from the Lévy continuity theorem and uniqueness theorem for characteristic functions it follows that

$$\int_{-\infty}^{\infty} e^{iuI} P_N(I)\, dI = \prod_{n=1}^{N} J_0(c_n u) = (J_0(c_0 u))^N, \tag{9}$$

where (Ohta, 1965)

$$P(I) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuI}(J_0(c_0u))^N \, du$$

$$= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuI} e^{-Ng(u)} \, du. \tag{10}$$

Here I denotes $I(t)$ ($= \lim_{N\to\infty} I_N(t)$) and

$$g(u) = -\log J_0(c_0u). \tag{11}$$

Consider a complex integral

$$\int_L e^{-sf(\zeta)}u(\zeta)\, d\zeta \qquad (\mathrm{Re}(sf(\zeta)) > 0). \tag{12}$$

The "method of steepest descent" due to P. Debye asserts that it is only a neighborhood of minimum value of $\mathrm{Re}(sf(\zeta))$ in the path of integration that makes an appreciable contribution to the integral (12) when $|s|$ is large. From the relation

$$g'(u) = J_1(c_0u)/J_0(c_0u) = 0, \tag{13}$$

values of u which give relative minima to $g(u)$ or zeros of $g'(u)$ are obtained as $u = 0, u_1, u_2, \ldots$ (Watson, 1922). However, $g(u)$ is found to attain an absolute minimum value at $u = 0$, i.e., $g(0) < g(u_i)$ ($i = 1, 2, \ldots$). In obtaining Eq. (13) an identity $dJ_0(u)/du = -J_1(u)$ has been used. In the light of the method of steepest descent mentioned above, it is clear that only a neighborhood of $u = 0$ makes an appreciable contribution to the integral (10) when N becomes large.

For small values of u, $J_0(c_0u)$ can be approximated as

$$J_0(c_0u) = 1 - \frac{(c_0u)^2}{4} + \frac{(c_0u)^4}{64} - \cdots \simeq \exp\left(-\frac{c_0^2}{4}u^2\right). \tag{14}$$

An autocorrelation function of $I_N(t)$ is given as

$$\Psi(\tau) = \langle I_N(t)\, I_N(t+\tau)\rangle = \sum_{n=1}^{N} \tfrac{1}{2}c_n^2 \cos 2\pi f_n\tau, \tag{15}$$

so we have $\Psi(0) = Nc_0^2/2$. The substitution of the last relation and Eq. (14) in Eq. (10) results in

$$P(I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuI} e^{-\frac{\Psi(0)}{2}u^2}\, du$$
$$= \frac{1}{\sqrt{2\pi\Psi(0)}} \exp\left(-\frac{(I-\langle I\rangle)^2}{2\Psi(0)}\right) \qquad (\langle I\rangle = 0). \qquad (16)$$

The above relation shows that $I_N(t)$ is asymptotically normal with mean zero and variance $\Psi(0)$ for large N. This completes the proof.

5. Experiment

First of all the new white noise model is simulated by digital computer according to Eq. (3) where we have set $c_n = 1$, $\phi_n = 0$ ($n = 1, 2,\ldots, N$) for convenience, and then it is examined by making use of chi-square tests, autocorrelation functions, scatter diagrams, and distribution curves to find out if it meets some important requirements of white noise.

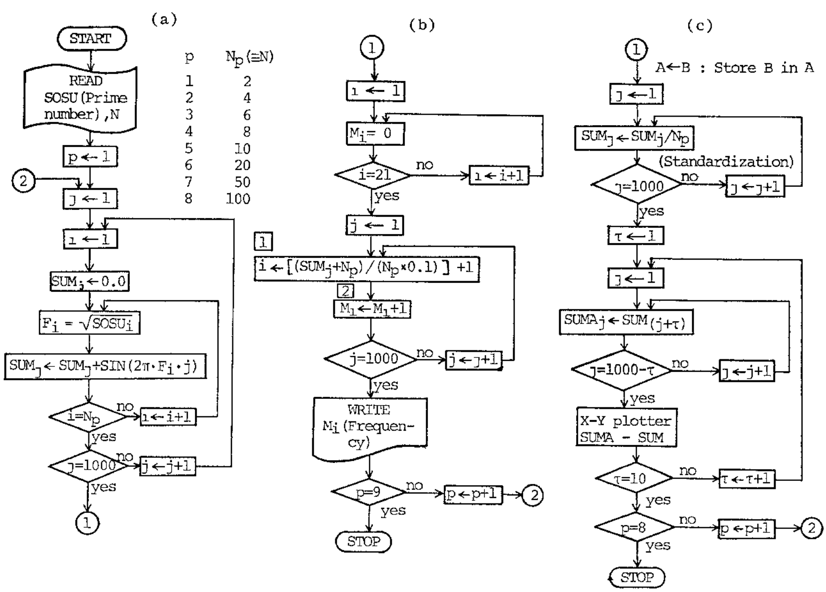

Fig. 1. Flow chart for digital simulation. (a) Flow chart for forming white noise; (b) flow chart for calculating values of frequency; (c) flow chart for drawing scatter diagrams.

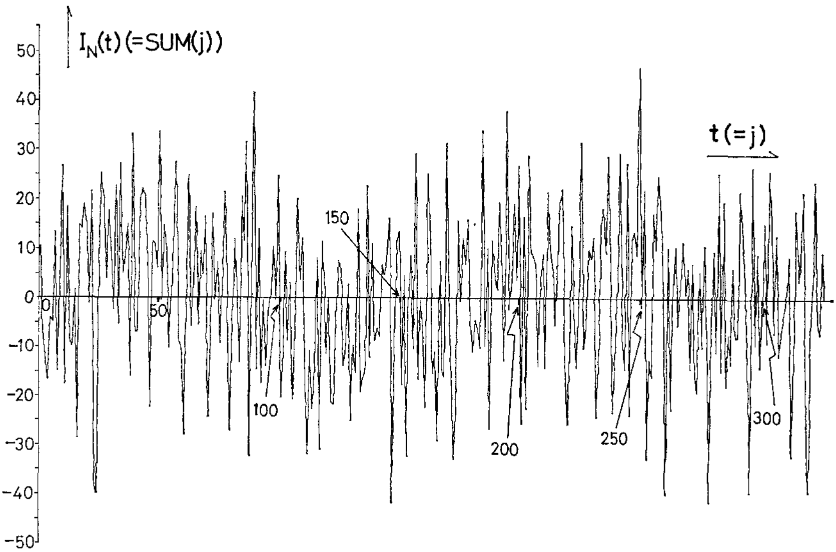

FIG. 2. A computer-simulated waveform of the new white noise model ($N = 50$).

Figure 1(a) is a flow chart for simulating white noise (cf. Fig. 2). SUM corresponds to $I_N(t)$ of Eq. (3). j, which corresponds to time t, varies from 1 to 1000. To obtain frequency ratios of irrational numbers, square roots (F_i) of prime numbers (2, 3, 5, 7, 11,...) have been chosen as f_n. We have chosen eight numbers, 2, 4, 6, 8, 10, 20, 50, 100, as the values of N. We use a variable integer P which varies consecutively from 1 to 8 to specify each value of N in the order mentioned. The computer is programmed to carry out all the required calculations for each value of N (= 2, 4, 6, 8,...) as P varies from 1 to 8.

The instantaneous value of $I_N(t)$ varies from $-N$ to N, as seen from Eq. (3) ($c_n = 1, \phi_n = 0$). The interval $[-N, N]$ is divided into a certain number of subintervals of equal length, each of which is designated a class. When the cumulative frequency distribution is calculated, we choose the upper end of a subinterval as a class mark for the corresponding class. The class of the instantaneous value SUM_j (at time j) is checked for each j in Block $\boxed{1}$ of Fig. 1(b). Every time SUM_j with magnitude belonging to the i-th class appears, it is summed up for each i in Block $\boxed{2}$ (cf. Table I). Figure 3 shows cumulative frequency distributions obtained in this way drawn on a normal probability paper. A standardized variable $Z = (X - \mu)/\sigma$ (X = value of level, μ = mean, σ = standard deviation) is used so that we

TABLE I

Values of Frequency Obtained by Means of Digital Simulation

i \ N	2	6	10	20	50
1	33	0	0	0	0
2	26	23	0	0	0
3	38	28	1	0	0
4	41	36	0	0	0
5	34	29	7	1	0
6	44	44	28	4	0
7	50	55	49	17	2
8	63	84	96	69	24
9	58	93	138	170	118
10	91	95	172	243	345
11	100	88	180	230	350
12	70	105	158	147	145
13	59	81	88	95	16
14	43	65	57	18	0
15	53	37	18	6	0
16	43	45	10	0	0
17	46	31	1	0	0
18	29	33	0	0	0
19	45	28	1	0	0
20	34	0	0	0	0
21	0	0	0	0	0

N = the number of component waves.
i = class number.

can readily compare the curves with the normal distribution curve. We can notice that the experimental curves approach the normal distribution curve as N becomes large. In Table II the result of a chi-square test for $N = 50$ is given. From a table of χ^2 distributions with 12 degrees of freedom, a value of p for $\chi^2 = 7.757$ is found to be somewhere between 0.80 and 0.90. This implies that we may accept the hypothesis that the simulated noise has a normal distribution even on an 80% level of significance.

In Fig. 4 is given a graphical representation of autocorrelation coefficients R_τ for τ ranging from 1 to 50,

$$R_\tau = \frac{\sum_{t=1}^{1000} (I_N(t) - \overline{I_N(t)})(I_N(t-\tau) - \overline{I_N(t-\tau)})}{\sqrt{\sum_{t=1}^{1000} (I_N(t) - \overline{I_N(t)})^2 \cdot \sum_{t=1}^{1000} (I_N(t-\tau) - \overline{I_N(t-\tau)})^2}}.$$

Notice that R_τ rapidly approaches zero as N becomes large, even when $\tau = 5$. Figures 5, 6, and 7 are scatter diagrams of the correlation between $I_N(t)$ and $I_N(t+\tau)$ plotted by an X–Y plotter (flow chart is given in Fig. 1(c)).

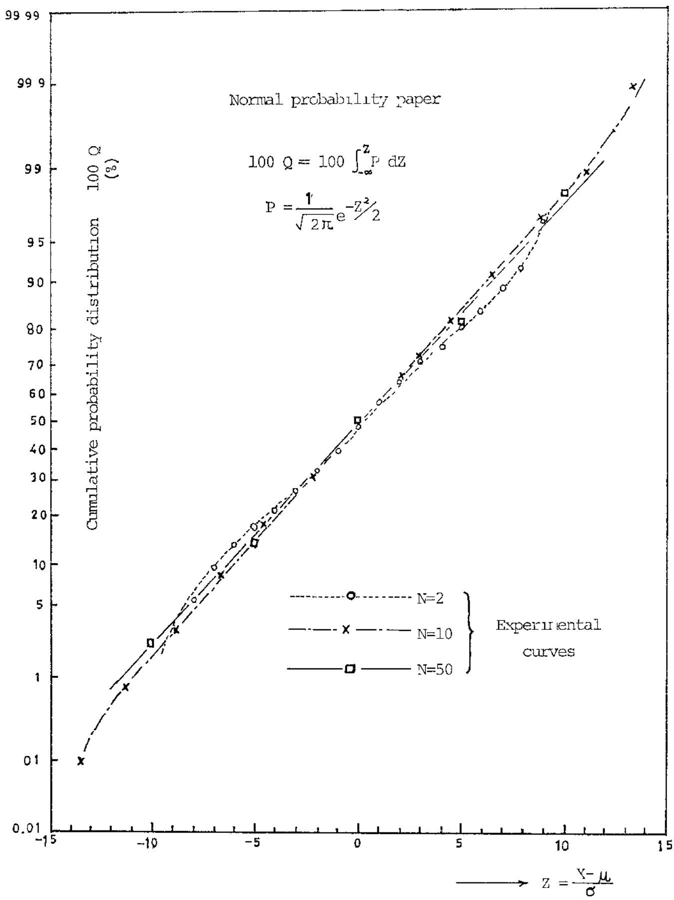

FIG. 3. Cumulative probability distribution of simulated white noise.

When $N = 2, 10$, the existence of correlation can be recognized for both $\tau = 1$ and 10, but when $N = 50$, no trace of correlation is observable even for $\tau = 1$.

TABLE II

Chi-Square (χ^2) Test Used for Testing the Closeness of Fit

i	The i-th class	Frequency f_i		Probability P_i	nP_i ($n = \sum_i f_i$)		$(f_i - nP_i)^2/nP_i$
1	—16.0 ~ —14.01	2	8	0.00249	2.49	8.13	0.002
2	—14.0 ~ —12.01	6		0.00564	5.64		
3	—12.0 ~ —10.01	17		0.01455	14.55		0.383
4	—10.0 ~ —8.01	30		0.03205	32.05		0.131
5	—8.0 ~ —6.01	59		0.06270	62.70		0.218
6	—6.0 ~ —4.01	90		0.09679	96.79		0.476
7	—4.0 ~ —2.01	125		0.13272	132.72		0.449
8	—2.0 ~ —0.01	157		0.15542	155.42		0.014
9	0.0 ~ 1.99	149		0.15542	155.42		0.265
10	2.0 ~ 3.99	152		0.13272	132.72		2.775
11	4.0 ~ 5.99	96		0.09679	96.79		0.006
12	6.0 ~ 7.99	63		0.06270	62.70		0.001
13	8.0 ~ 9.99	38		0.03205	32.05		1.07
14	10.0 ~ 11.99	12	16	0.01455	14.55	22.68	1.967
15	12.0 ~ 13.99	3		0.00564	5.64		
16	14.0 ~ 15.99	1		0.00249	2.49		

$$\chi^2 = \sum_i (f_i - nP_i)^2/nP_i = 7.757$$

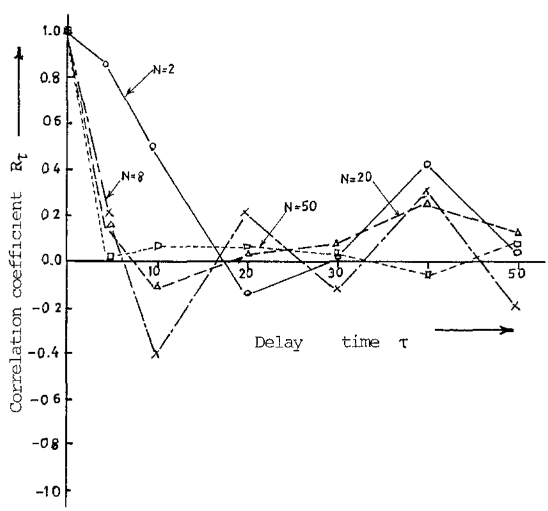

FIG. 4. Autocorrelation function of simulated white noise.

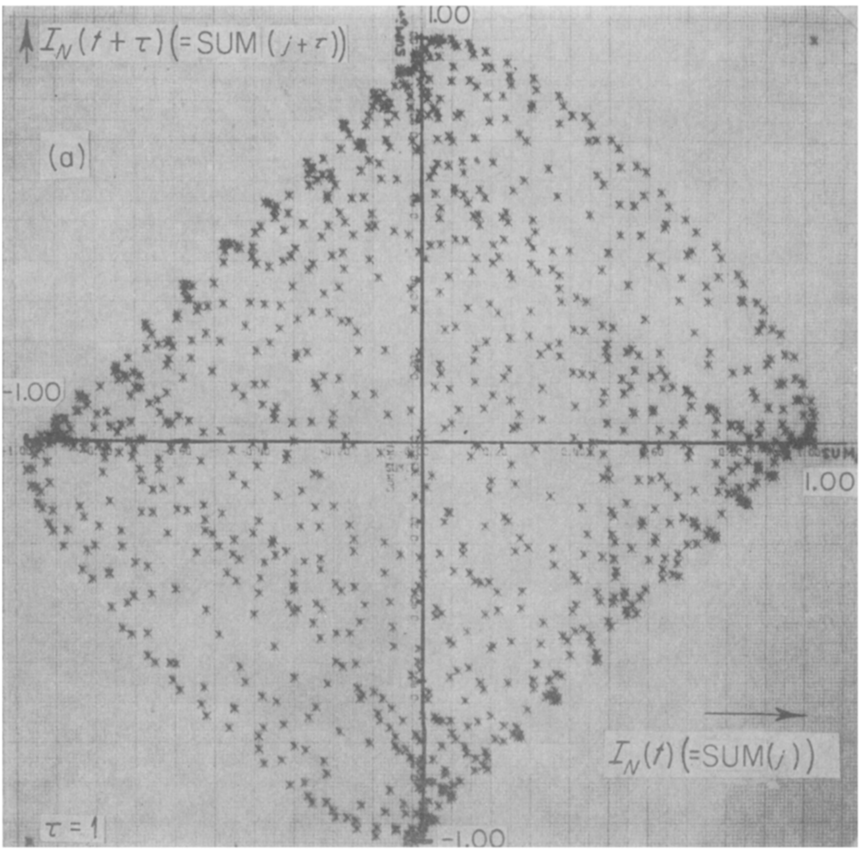

FIGURE 5

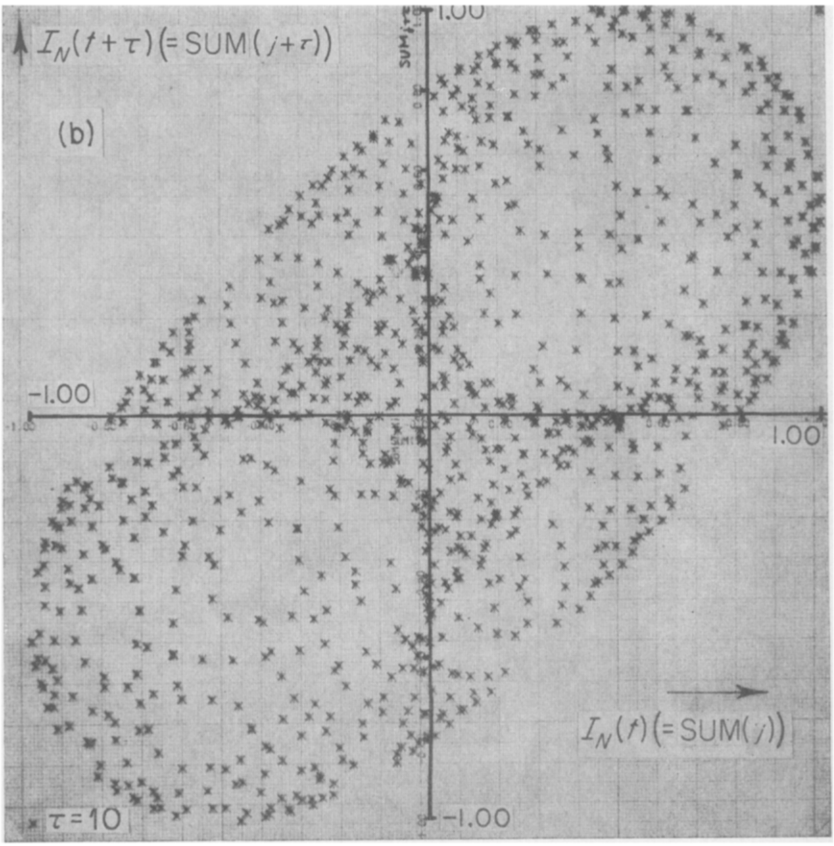

FIG. 5. Scatter diagrams of simulated white noise for the case $N = 2$ (positive correlation). (a) $\tau = 1$; (b) $\tau = 10$.

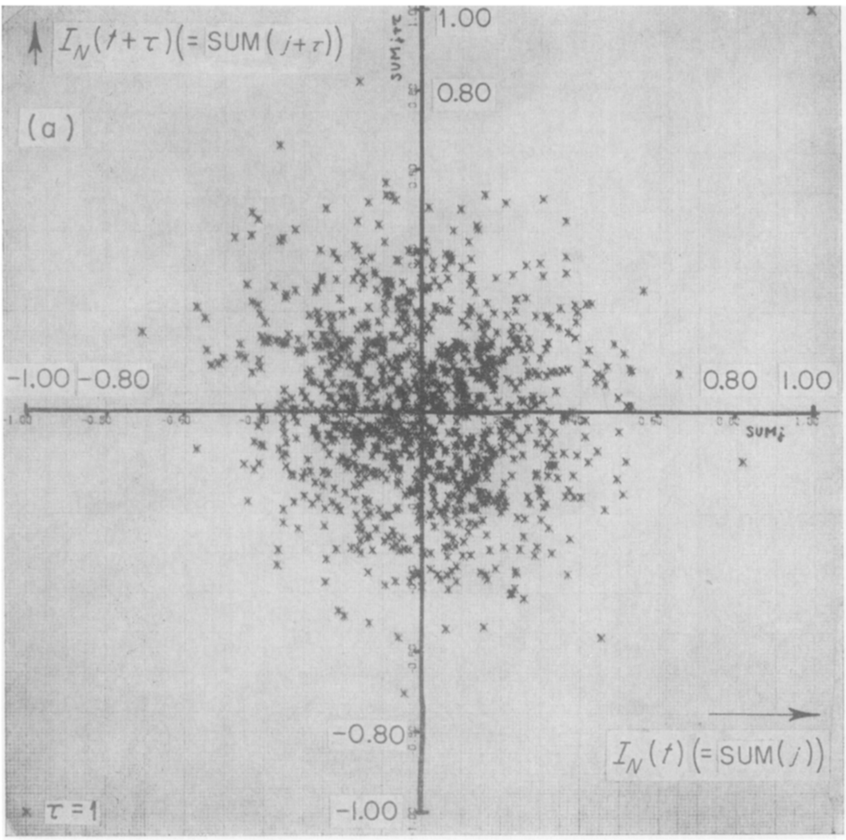
$I_N(t+\tau)$ (= SUM $(j+\tau)$)
(a)
1.00
0.80
-1.00 -0.80
0.80 1.00
SUM$_j$
-0.80
-1.00
$I_N(t)$ (= SUM(j))
$\tau = 1$

FIGURE 6

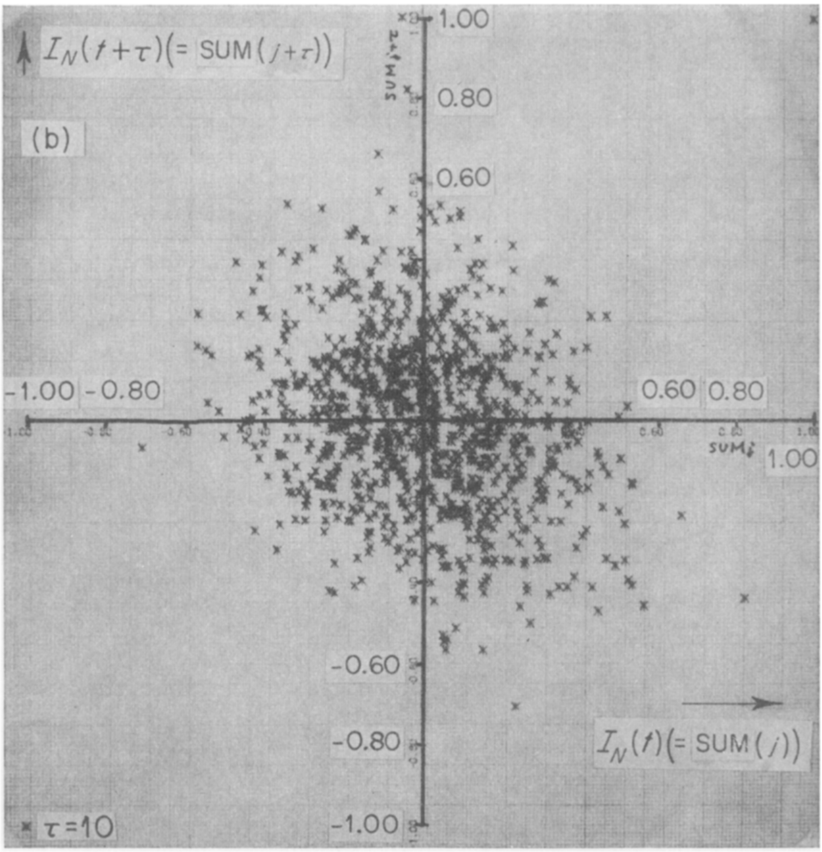

FIG. 6. Scatter diagrams of simulated white noise for the case $N = 10$ (slightly negative correlation). (a) $\tau = 1$; (b) $\tau = 10$.

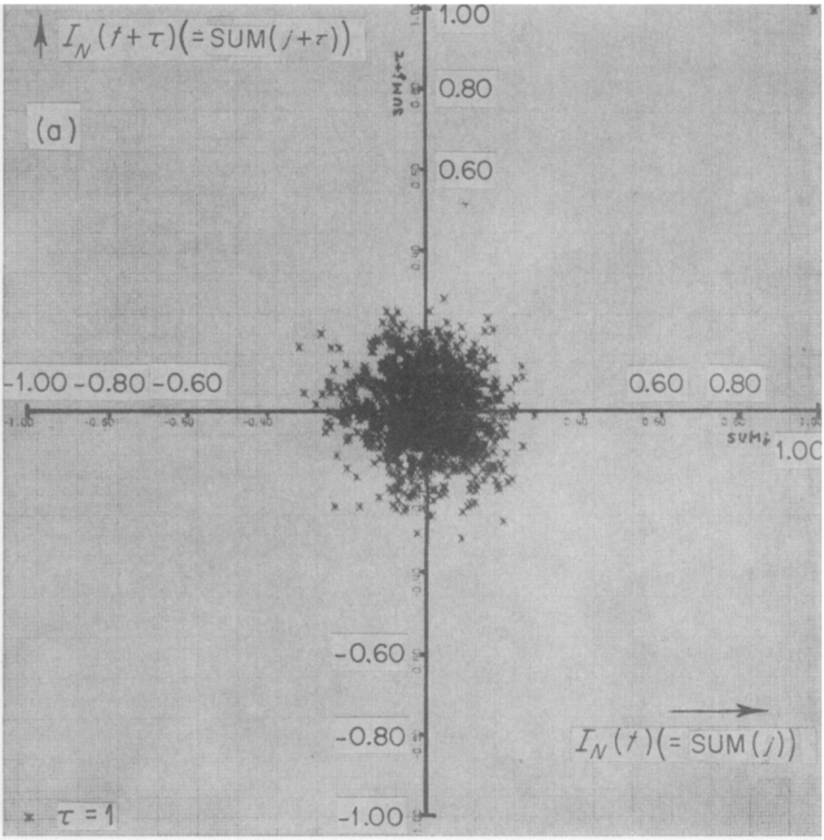
$I_N(t+\tau)$(= SUM($j+\tau$))
(a)
1.00
0.80
0.60
-1.00 -0.80 -0.60
0.60 0.80
1.00
-0.60
-0.80
-1.00
$I_N(t)$(= SUM(j))
$\tau = 1$

FIGURE 7

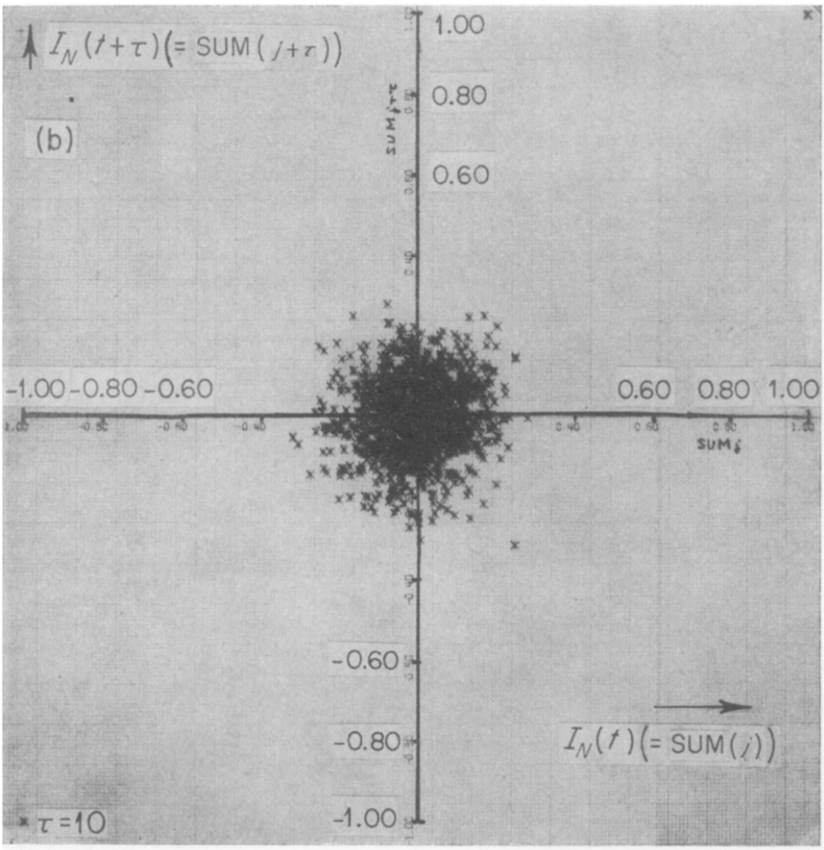

FIG. 7. Scatter diagrams of simulated white noise for the case $N = 50$ (almost no correlation). (a) $\tau = 1$; (b) $\tau = 10$.

6. Conclusion

The Rice representations are deterministic Fourier series representations of normal random noise in which amplitudes a_n, b_n and phases φ_n of Fourier components are assumed to follow normal and uniform distributions, respectively. First of all we have noticed that the probability distributions are brought into the representations independently of time to express possible variation of amplitude or phase at an arbitrary time, and then we have introduced the new mathematical model of white noise expressed in terms of u.a.p. functions which is characterized by

(1) No necessity of assuming probability distribution for either amplitude or phase, and

(2) Normal distribution characteristics shown in the course of time.

If white noise is regarded as the sum of outputs of infinitely many independent oscillators having different frequencies, it may be more natural to think that all the frequency ratios are irrational numbers than to think that they are rational numbers or integers, because the set of all irrational numbers is exceedingly dense compared with that of all rational numbers or integers and so a probability that the frequency ratios happen to be rational numbers or integers is much less than a probability that they are irrational numbers.

Therefore, the new model of white noise seems to explain the actual structure of random noise better than the Rice representations. Apart from this, it would not have been possible to derive the model without the assumption of the irrationality of frequency ratios.

However, the point discussed above, which may be connected with ergodicity, does not always guarantee the usefulness of the model in applications to engineering. The Rice representations of random noise are found to be of more practical significance in many situations, though in a strict sense they do not contain all real frequency components exhaustively.

Finally, by means of digital simulation it has been confirmed that the new mathematical model of random noise exhibits white noise characteristics for large N.

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